Enhancement of the downhill simplex method of optimization

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ABSTRACT

The downhill simplex method of optimization is a "geometric" method to achieve function minimization. The standard algorithm uses arbitrary values for the deterministic factors that describe the "movement" of the simplex in the merit space. While it is a robust method of optimization, it is relatively slow to converge to local minima. However, its stability and the lack of use of derivates make it useful for optical design optimization, especially for the field of illumination. This paper describes preliminary efforts of optimizing the performance of the simplex optimizer. This enhancement is accomplished by optimizing the various control factors: alpha (reflection), beta (contraction), and gamma (expansion). This effort is accomplished by investigating the "end game" of optimal design, i.e., the shape of the figure of merit space is parabolic in *N*-dimensions near local minima. The figure of merit for the control factor optimization is the number of iterations to achieve a solution in comparison to the same case using the standard control factors. This optimization is done for parabolic wells of order $N = 2$ to 15. In this study it is shown that with the correct choice of the control factors, one can achieve up to a 35% improvement in convergence. Techniques using gradient weighting and the inclusion of additional control factors are proposed.

OCIS codes: (080.2720) Geometrical optics, mathematical methods, (000.3860) Mathematical methods for physics, (220.2740) Geometrical optics, optical design.

1. Introduction

Optical design and analysis makes prevalent use of optimization techniques to improve the performance of an optical system. The sub-field of illumination design is increasingly requiring optimization in order to achieve system demands. There is a variety of optimization methods, both local and global, used in optical design packages: simulated annealing, damped least squares, simplex, genetic selection, and so forth. The choice of the optimization method is dependent on the problem being solved, but the downhill simplex method of optimization provides a simple, self-contained method that works with any order dimensional space. It is especially useful for illumination design optimization for two primary reasons: (1) stability of the algorithm is assured for the long process times inherent in illumination system analysis and (2) the merit space for illumination system optimization is not well defined (i.e., there is limited ability in using methods that employ derivatives). The simplex algorithm gives a robust method that does not rely on derivatives to provide function minimization for any order dimensional space, but at the expense of convergence speed due to the increased number of function calls in comparison to other optimization algorithms. Speed of convergence is especially limited as the dimensionality of the merit space is increased. A previous general rule of thumb was that dimensionality was limited to around 10 to 15 variables, which of course limited the utility of simplex in optical design. However, with the advent of faster computer processors, the dimensional limitation is perceived to be reduced. Simplex, while still having a slower convergence speed than other algorithms, can be used in a manner conducive to obtaining timely results.

The downhill simplex method of optimization uses a "geometric" construct, called a simplex, to achieve function optimization (i.e., minimization). In a sense the simplex "rolls" downhill due to computation of the function values at the vertices of the simplex, replacing vertices (except the low value) within each iteration of the algorithm. The basis of the algorithm was developed in 1962 [1], but in 1965 Nelder and Mead are attributed with developing its modern form [2]. The implementation remains essentially unchanged since Nelder and Mead, and can be found in several literature sources (see for example Ref. [3]). The technique uses three factors to "move" the simplex through the merit space: alpha (reflection moves), beta (contraction moves), and gamma (expansion moves). The three factors were given arbitrary values of 1.0, 0.5, and 2.0 respectively. As discussed previously the technique, while being robust, has a limited convergence speed in comparison to other optimization methods. Several authors have considered the convergence speed while improving upon the robustness of the algorithm, including: false convergence within the

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contraction moves [4], selection of the initial simplex [5], restarting of the procedure to find true local minima [6], and selection of different factors dependent on the dimensionality of the function [7]. Of these, Refs. [4] and [7] address the arbitrariness of the selection of the factors used in the Nelder and Mead simplex method. They investigate using different values for the factors, especially the contraction term in order to speed up convergence. Their selection of modified factor values is once again arbitrary, thus the choice of the factors is likely non-optimal. Additionally, these two references break down the two possible contraction steps into two separate factors: contraction in one dimension and contraction around the current low figure of merit value. Using one and two contraction factors is presented herein.

While the simplex method has benefited from faster processors, improvements to the convergence speed are both useful and possible. This enhanced performance bodes well for the applicability to optical, especially illumination, system design. A technique based on using optimization to actually improve the simplex optimization method is investigated herein: the standard Nelder and Mead algorithm is used to optimize the factors in an altered simplex routine. The function being minimized in the altered simplex is arbitrary, so I use an N-dimensional parabolic well. A parabolic well mirrors the shape of the merit space near a local minimum. This choice thus represents the "end game" in function minimization, which requires the dominant amount of time during optimization (i.e., the simplex "falls" into the well and converges to a solution by contracting). In the next section an overview of the simplex method is provided. Following this section, the optimizer to improve the performance of simplex is presented. Next the results for N = 2 - 15 are presented. Finally, conclusions and future research efforts are discussed

2. Overview of the Nelder and Mead simplex method

The simplex method in N dimensions uses N+1 points within the merit space to define the simplex. Selection of these points can be prescribed, but random selection allows the potential to fully investigate the merit space. The function values are found at each of these points. The points with the low (P_L), high (P_1), and second high (P_2) function values are determined. Next, the centroid of the points except P_1, $\overline{P}$, is determined. The simplex method essentially has four steps possible during each iteration: reflection, contraction in one dimension, contraction around the low vertex, and expansion. The basis for each step is provided here:

(1) Reflection: A reflected point, P_R, is found by reflecting P_1 through $\overline{P}$ with the equation

$$P_R = (1+\alpha)\overline{P} - \alpha P_1, \tag{1}$$

where α is the reflection factor (Nelder and Mead: $\alpha = 1$). P_R replaces P_1 if $f(P_L) < f(P_R) < f(P_1)$.

(2) Expansion: if $f(P_R) < f(P_L)$ then the simplex grows along the centroid direction with the hope that the expansion point, P_E, is better than P_L. The expansion is determined with the equation

$$P_E = (1-\gamma)\overline{P} + \gamma P_R, \tag{2}$$

where γ is the expansion factor (Nelder and Mead: $\gamma = 2$). P_E replaces P_1 if $f(P_E) < f(P_L)$.

(3) 1D Contraction: if $f(P_R) > f(P_2)$ then the simplex contracts along the centroid direction with the hope that the contracted point, P_C, is better than P_2. The 1D contraction is determined with the equation

$$P_C = (1-\beta_1)\overline{P} + \beta_1 P_0, \tag{3}$$

where β_1 is the 1D contraction factor (Nelder and Mead: $\beta_1 = 0.5$) and P_0 is the selection of P_1 or P_R which has the lowest function value. P_C replaces P_1 if $f(P_C) < f(P_0)$.

(4) Full contraction: if $f(P_C) > f(P_0)$ then 1D contraction does not suffice, and the whole simplex is contracted around P_L. The full contraction is determined with the equation

$$P_i = (1-\beta_2)P_L + \beta_2 P_i, \tag{4}$$

where β_2 is the full contraction factor (Nelder and Mead: β_2= 0.5) and P_i represents all the points except P_L.

Typically, when a point replaces P_1 the current iteration is completed. Next the termination condition is checked. If the tolerance is not met then the next iteration is started. If the tolerance is met, then the optimization is done. The termination condition as per Nelder and Mead is the standard deviation of the figure of merit values. Alternate stopping conditions are developed in the next section. The reader is encouraged to consult Refs. [2] and [3] for further description of the standard simplex method. Note that small modifications have been made to the Nelder and Mead algorithm in order to improve performance. These modifications are based on Ref. [4] and primarily affect the 1D contraction.

3. Description of the optimizer to improve simplex performance

An optimizer was built around the simplex algorithm that is minimizing a simplex within an N-dimensional parabolic well. Any optimizer can be used, but in order to remain consistent, the standard downhill simplex routine as described in Section 2 is used. Therefore, I am using a simplex optimizer to optimize the factor values of a second simplex that is finding the minimum in an N-dimensional parabolic well. A flowchart depicting the algorithm is shown in Fig. 1.

During the development of this algorithm many items had to be addressed to provide a working solution:

(1) Initial factor simplex: the α values are between 0 and 2, the γ values are greater than 1 and α, and two choices for the contraction factors: (a) $\beta_1 = \beta_2 = \beta$ with β less than 1 and (b) β_1 and β_2 are both randomly chosen less than 1. No constraints are placed on the factor simplex values during the iterations except they remain positive.

(2) Initial parabolic simplex: the points for the parabolic simplex are randomly chosen in a symmetric region around the minimum ($\mathbf{r} = \mathbf{0}$). No constraints are placed on the parabolic simplex during the iterations.

(3) Number of parabolic runs: for each case with a prescribed set of factor values for the simplex, the parabolic optimization was performed 1000 times (hereafter labeled as n_p). This arbitrary number of runs ensures that stochastic noise due to selection of the random parabolic simplex does not provide erroneous results. The figure of merit (*FOM*), described later, is then the average value of all of the runs. Each of the runs uses a randomly generated parabolic simplex as described in (2).

(4) False convergence correction: during initial development of the complete routine it was noted that convergence to non-minimal parabolic simplex values would occur. This phenomenon is speculated to have occurred due to the selection of the factor simplex values. In other words, the deviation of the factor simplex values from the standard simplex values would result in degenerate solutions located on the sloped surface of the parabolic well (i.e., greater the tolerance bounds). The termination condition (standard deviation of the function values in this case) was met, so premature stoppage would occur. To correct this obvious error and since the location of the minimum was known in advance a multiplicative factor of $1+\bar{f}$ was included in the parabolic simplex *FOM*, where $\bar{f}$ is the average functional value of the vertices of the final parabolic simplex.

(5) Number of factor runs: due to false convergence and the noted number of local minima in factor simplex space, up to 25 runs (hereafter n_f) were done per each N-dimensional well. During this investigation N = 2 - 15.

(6) Normalization of factor *FOM*: in order to compare effectively the altered and standard factors cases, each run (see item 3) was run with the same initial parabolic simplex for both the altered and standard factors cases.

(7) Factor simplex termination condition: it was noted during initial development that, while highly unlikely, premature stoppage of the factor simplex could occur if the functional standard deviation was within tolerances even though the simplex was still large. This phenomenon is likely more pronounced if the number of parabolic runs is reduced (see item 3). To counteract this unlikely event, the stopping condition was the size of the simplex. Once the size of the simplex is below tolerances, a prescribed number of additional iterations (e.g., 20) are done to ensure that convergence has been met.

(8) Parabolic simplex termination condition: the standard deviation of the functional values was used. Once this standard deviation is within tolerances the parabolic simplex is stopped.

(9) Parabolic simplex *FOM*: the parabolic simplex uses the following as its figure of merit (FOM)

$$FOM_{par} = M\left(1+\bar{f}\right), \tag{5}$$

where M is the number of iterations to convergence.

(10) Factor simplex *FOM*: the factor simplex uses the following as its FOM

$$FOM_{fact} = \frac{1}{n_p}\sum_{i=1}^{n_p}\frac{FOM_{par,altered}}{FOM_{par,std}} \tag{6}$$

where n_p is the number of runs for each parabolic optimization (i.e., $n_p = 1000$), the altered *FOM* is the result using Eq. (5) and the altered simplex factors, and the std *FOM* is the result using Eq. (5) and the Nelder and Mead factors. Thus, FOM_{fact} is the ratio of the altered factor simplex to the standard factor simplex. A value less than one indicates a solution for the altered values that improves convergence speed.

In the next section the algorithm developed in this section is tested on parabolic wells of order $N = 2$ - 15.

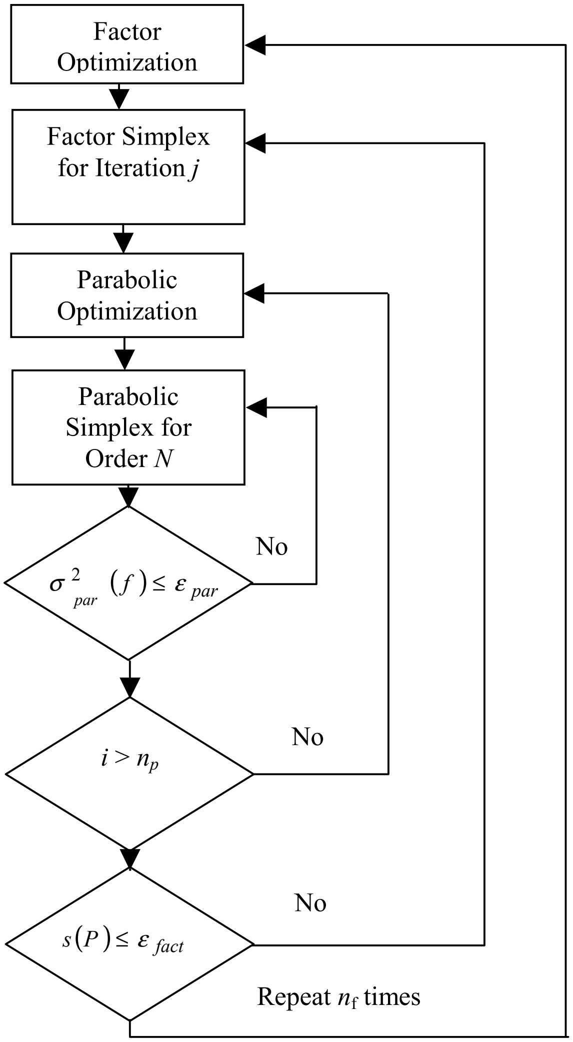

Fig. 1. Flowchart showing the procedure to find the optimal control factors for an N-dimensional parabolic well.

4. Results

4.1 Simplex Optimization Results

In Table 1 are the results for the optimization of an N-dimensional parabolic well (N up to 15) when a single contraction factor (β) is used. Table 2 shows the analogous results when there are two contraction factors. The first column is the order of the investigation. The next three or four columns for Tables 1 and 2 respectively provide the local optimum for the control factors. The additional columns indicate the convergence factor ratio (FOM_{fact}), the percentage of n_f false convergence solutions, and the percentage of the n_f solutions (not including false convergence ones) that converged slower than the simplex with the standard Nelder and Mead control factors, respectively.

Table 1. Control factor values and resulting data for the simplex optimization with one contraction factor.

N	α	β	γ	FOM_{fact}	False (%)	Slow (%)
2	1.00201847	0.33708846	2.14047762	0.787	4.0%	0.0%
3	1.00584013	0.40437032	2.0318591	0.916	12.0%	0.0%
4	1.00453446	0.45350011	1.76558873	0.986	0.0%	24.0%
5	1.00973246	0.49607063	1.57854835	0.992	15.0%	50.0%
6	1.00901577	0.51837181	1.5920392	0.967	4.0%	16.0%
7	1.02057426	0.55478902	1.41451365	0.924	8.0%	4.0%
8	1.0450278	0.58799849	1.384863	0.875	7.7%	0.0%
9	1.05981334	0.58479747	1.33104459	0.823	25.0%	0.0%
10	1.06231181	0.62128139	1.38151511	0.771	0.0%	0.0%
11	1.05448852	0.62818877	1.45295772	0.725	25.0%	0.0%
12	1.05181909	0.64176472	1.33758373	0.678	0.0%	0.0%
13	1.10417521	0.6514322	1.23423103	0.640	0.0%	25.0%
14	1.14479874	0.66084722	1.0160041	0.648	0.0%	0.0%
15	0.75153417	0.62507252	0.97600608	0.505	0.0%	0.0%

Table 2. Control factor values and resulting data for the simplex optimization with two contraction factors.

N	α	β_1	γ	β_2	FOM_{fact}	False (%)	Slow (%)
2	1.02163936	0.3400907	1.97470579	0.15928282	0.788	0.0%	0.0%
3	1.0047228	0.40248282	2.01219185	0.63582727	0.922	0.0%	0.0%
4	1.00764764	0.47145135	1.74322692	0.57732618	0.985	8.0%	16.0%
5	0.99262641	0.4937762	1.48825745	0.52602173	0.995	24.0%	32.0%
6	1.01666875	0.52542045	1.49081084	0.63782812	0.970	0.0%	5.0%
7	1.02666482	0.54712469	1.41155468	0.6668768	0.926	5.0%	0.0%
8	1.05871964	0.56811415	1.42073806	0.75380829	0.877	0.0%	0.0%
9	1.05786796	0.61070155	1.44480667	0.9669131	0.826	21.4%	0.0%
10	1.05880265	0.60468934	1.35682635	0.30978598	0.766	25.0%	12.5%
11	1.08637866	0.63946823	1.4208826	0.53437302	0.724	20.0%	0.0%
12	1.07848799	0.64421344	1.40881131	0.53868179	0.674	0.0%	0.0%
13	1.076295	0.64196353	1.42068615	0.41292272	0.632	0.0%	0.0%
14	0.76023269	0.69658156	1.0641866	0.14172604	0.584	0.0%	0.0%
15	0.76135727	0.67405069	1.35150713	0.09221549	0.513	0.0%	0.0%

First, most of the optimized factor values tend to follow a trend for both cases, except for $N = 15$ in Table 1 and $N = 14$ and 15 for Table 2. The specific trends are:

1. The reflection factor (α) tends to increase with the dimensionality of the parabolic well (N) as shown in Fig. 2. Adding the second contraction factor (i.e., Table 2) played a minor role in the determination of the reflection factor. The difference between the two cases is likely due to neighboring local minima. The increase is not constant, but the tendency is to increase due to discrete jumps. At $N = 5$ the value of the standard Nelder and Mead algorithm is obtained. The jumps are likely due to the increased number of local minima as N increases.
2. The 1D contraction factor (β_1) increases with the dimensionality of the parabolic well (N) as shown in Fig. 3. In the two contraction factor cases, the 1D contraction values are nearly equal. The increase is steady with $N = 5$ obtaining the value of the standard Nelder and Mead algorithm.
3. The expansion factor (γ) tends to decrease with the dimensionality of the parabolic well (N) as shown in Fig. 4. In the two cases shown, the expansion values are nearly equal. The decrease is steady with $N = 3$ providing the value of the standard Nelder and Mead algorithm.
4. The full contraction factor (β_2) tends to be variable as a function of the dimensionality of the parabolic well (N) for the two-contraction factor case as shown in Fig. 5. For the one-contraction factor case the single contraction factor case (see point 2 above) is shown. The randomness of the two-contraction factor case indicates the presence of local minima.

Second, as expected, FOM_{fact} does not change appreciably between the two cases for a given dimension order, which is shown in Fig. 6. This result is due to little use of the full contraction factor (β_2) during the optimization. At $N = 5$ the FOM_{fact} is just below 1.0 indicating that the standard Nelder and Mead algorithm is optimized for an end-game solution with 5 variables. Away from N = 5 the speed of convergence is increased over the standard Nelder and Mead method. This result indicates that the values for the optimization should be set dependent on the number of variables, especially when it has been determined that the optimization process is within the end-game window.

Third, there are a number of nuances that can be seen with closer inspection of the data. As can be seen from the tables and figures there is little difference in the reflection factor as N changes. In fact there is little change in α from the standard Nelder and Mead value of 1.0. The slightly larger reflection value indicates that the speed of convergence is slightly improved with a larger value. On the other hand (β_1, β_2) and γ increase and decrease respectively with a larger range of change as N increases. The value of the β_1 passes the standard Nelder and Mead value of 0.5 at $N = 5$, while β_2 appears somewhat random but almost always greater than the standard Nelder and Mead value of 0.5. The smaller 1D contraction factor for $N < 5$ indicates that shrinkage of the simplex on an individual iteration must not be too great to cause premature stoppage. In higher dimensions, the simplex is spending the bulk of its "time" shrinking; therefore, a larger shrinkage factor leads to convergence with less iterations. The expansion term, γ, is less than the Nelder and Mead value 2.0 after $N = 3$. This smaller value indicates that in the end game, the simplex benefits from minimizing expansion since contraction is the primary aspect of this phase of the optimization process. The expansion term is used in the "start game" to find a local minimum valley, while the contraction term is used to find the minimum in this valley. Finally, as provided in the two tables the False percentage, and the Slow percentage increase until $N = 5$. From the complete set of data runs per dimension (not provided herein), the false convergence cases arise when α is less than 1.0 (i.e., approaching 0.5), thus the reflection introduces a contraction at the same time. The increase in the Slow percentage is due to two reasons: FOM_{fact} approaches 1.0 as N increases or the value of γ is less than that of α. In the next section the values of Table 2 are used in the optimization of an illumination system design.

In Figs. 7 – 10 are shown the values for the expansion, 1D contraction, expansion, and full contraction respectively as a function of FOM_{fact} for the two-contraction factor case. The results are shown for $N = 2$, 3, 12, and 15. Thus, the plots show part of the figure of merit space for these four dimensions. Similar results can be shown for the other dimensions of parabolic wells. Figure 7 indicates that the reflection merit space is fairly flat for low orders but "peaked" for higher orders. Figure 8 indicates that the 1D contraction merit space is antithetical to that of Fig. 7, or in other words it is "peaked" for lower orders and flat for higher orders. Figure 9 shows that the expansion merit space is fairly flat for all orders; however, there is a distinctive "tilt" indicating smaller values are preferred. Finally, the full contraction merit space is flat over the whole extent with the evident variability due to false convergence cases.

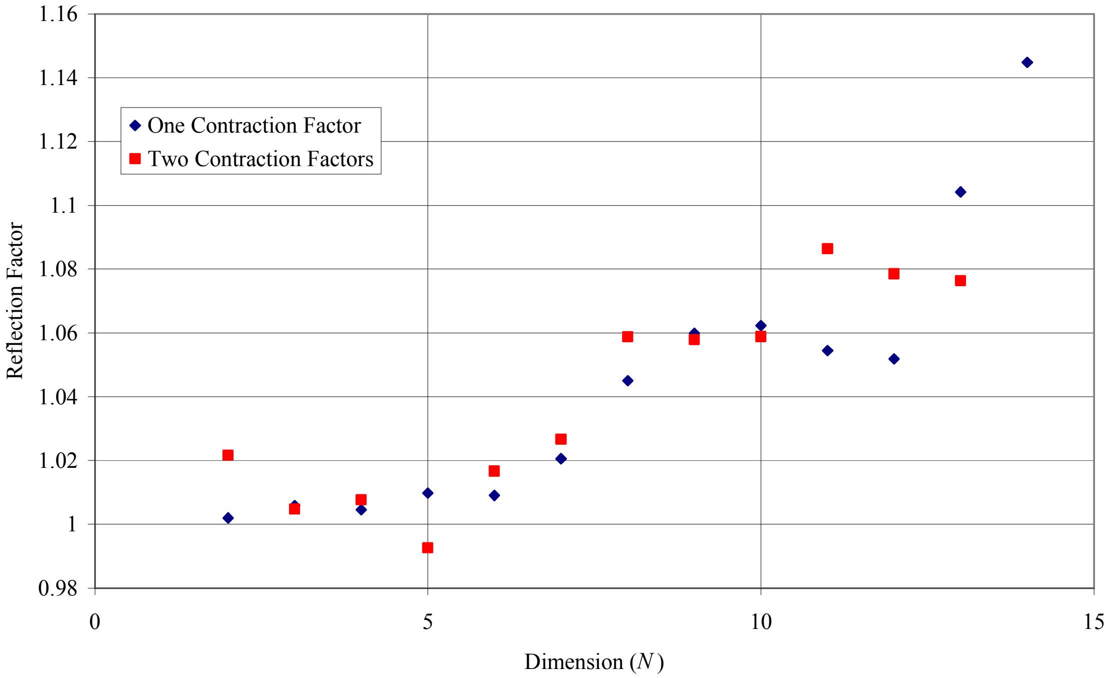

Figure 2. Reflection factor (α) for one (“diamonds”) and two (“squares”) contraction factors as a function of dimensionality of the parabolic well (N).

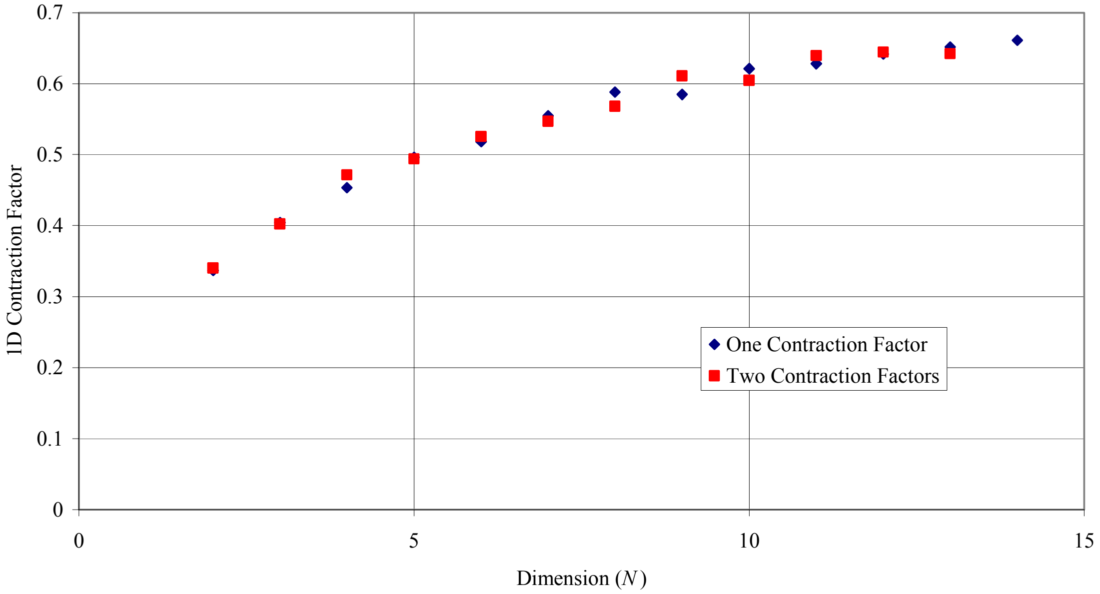

Figure 3. 1D contraction factor (β_1) for one (“diamonds”) and two (“squares”) contraction factors as a function of dimensionality of the parabolic well (N).

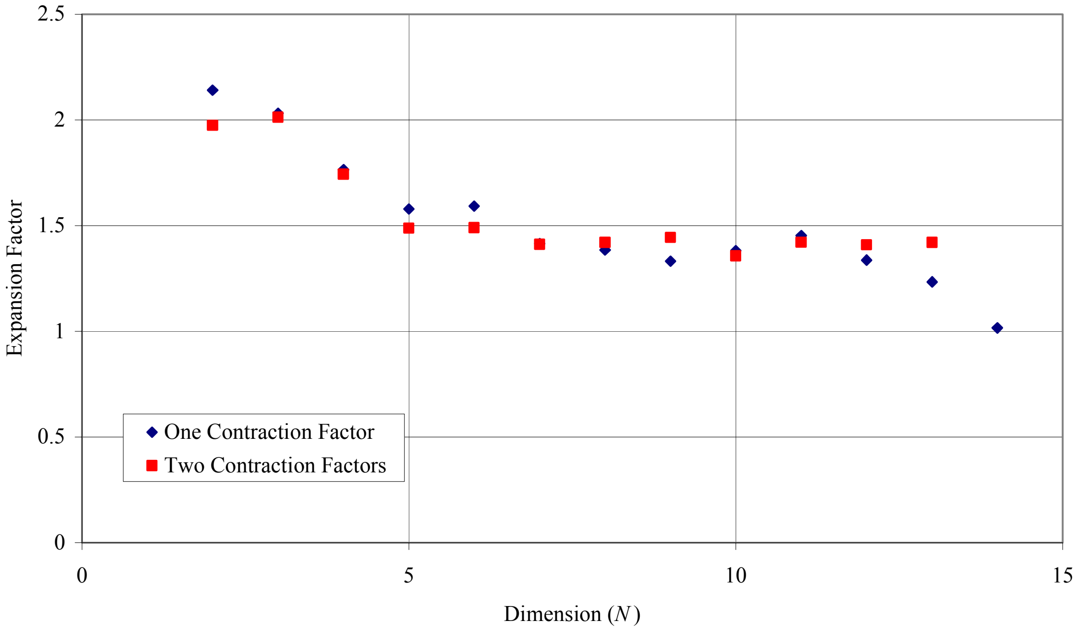

Figure 4. Expansion factor (γ) for one ("diamonds") and two ("squares") contraction factors as a function of dimensionality of the parabolic well (N).

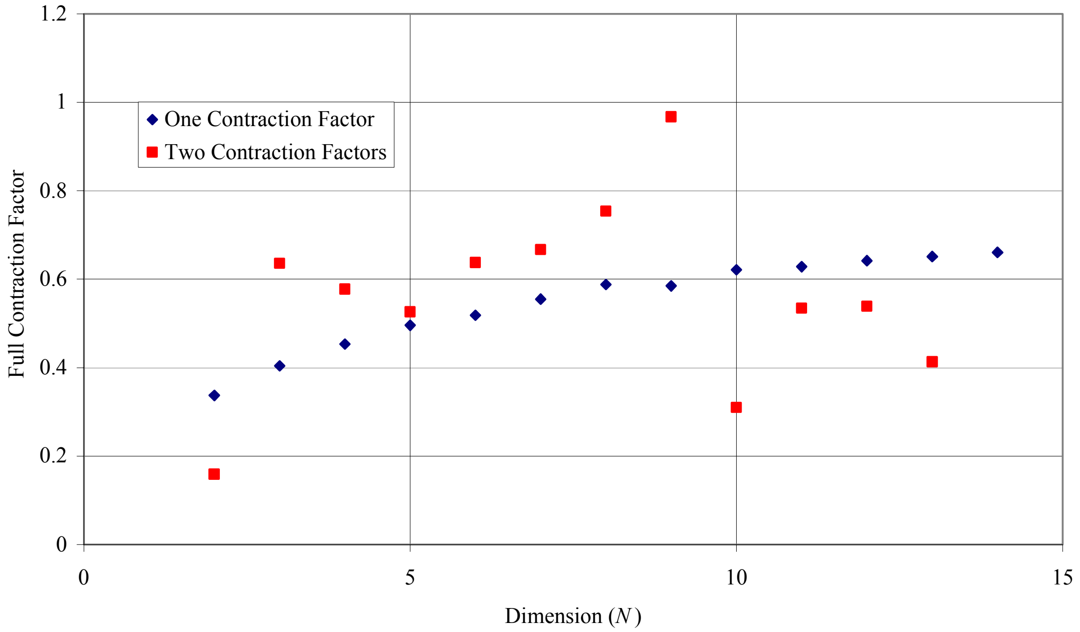

Figure 5. Full contraction factor (β_2) for one ("diamonds") and two ("squares") contraction factors as a function of dimensionality of the parabolic well (N).

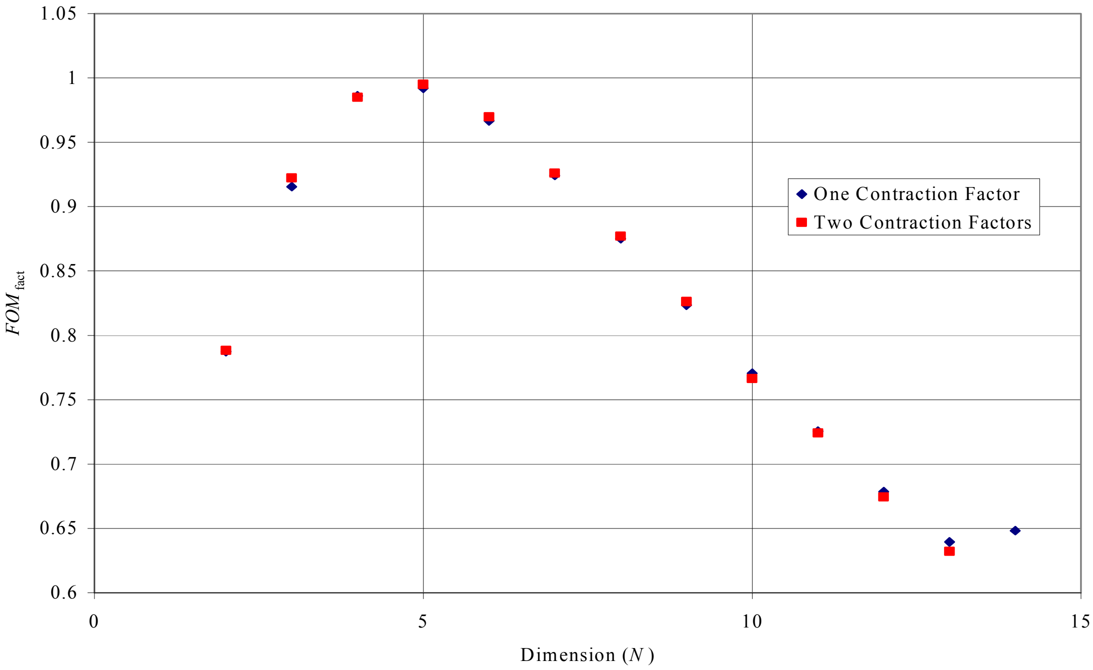

Figure 6. FOM_{fact} for one (“diamonds”) and two (“squares”) contraction factors as a function of parabolic well dimensionality (N).

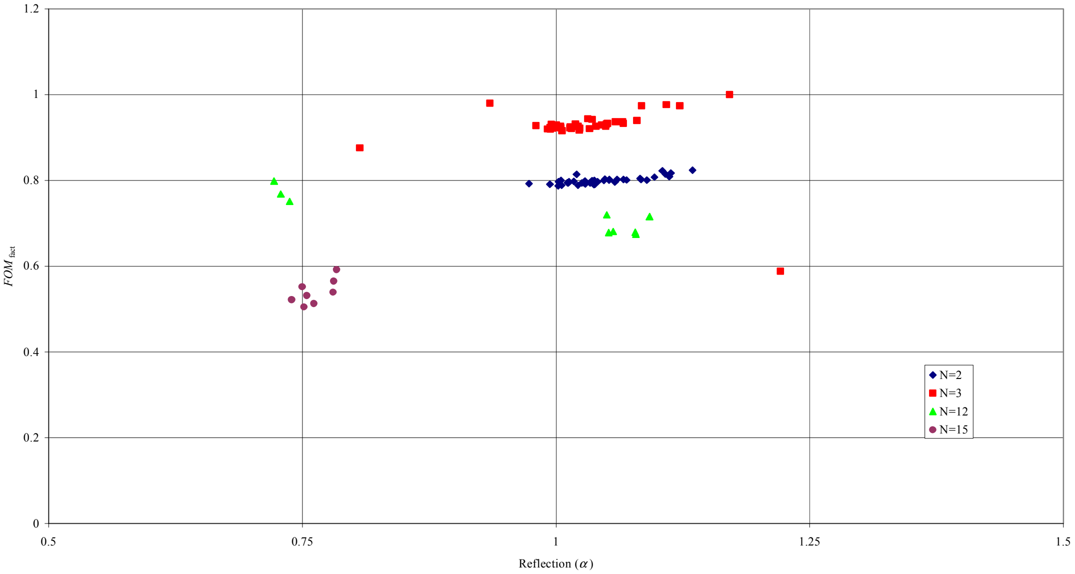

Figure 7. FOM_{fact} as a function of the optimized reflection values for the two-contraction factor case. The results for the many data runs are shown for $N = 2$, 3, 12, and 15. Similar results are obtained for the other dimensions.

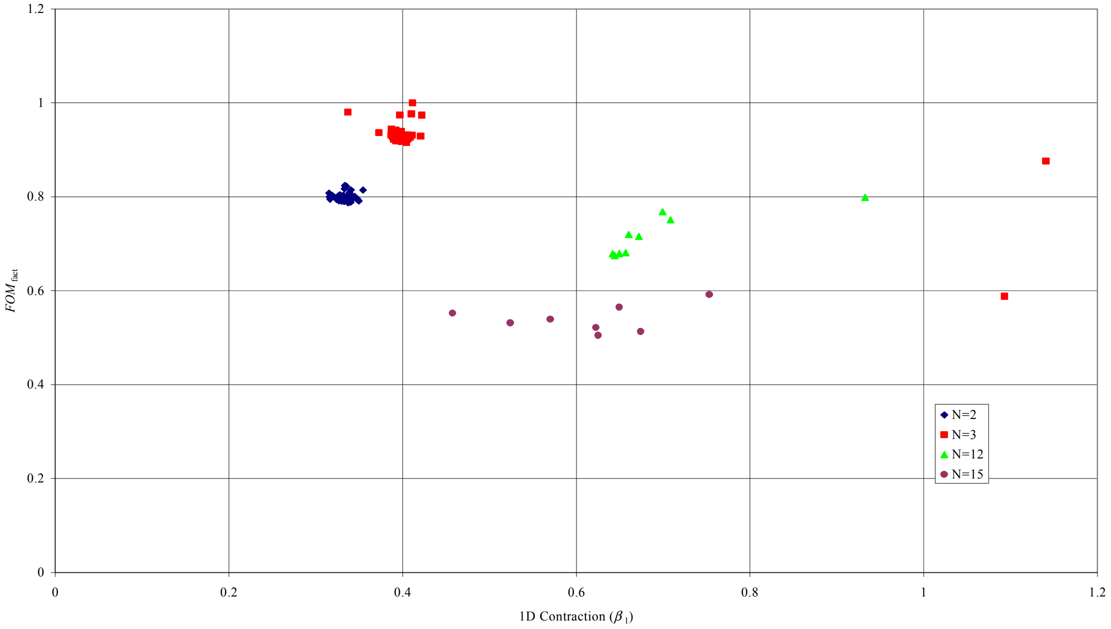

Figure 8. FOM_{fact} as a function of the optimized 1D contraction values for the two-contraction factor case. The results for the many data runs are shown for $N = 2$, 3, 12, and 15. Similar results are obtained for the other dimensions.

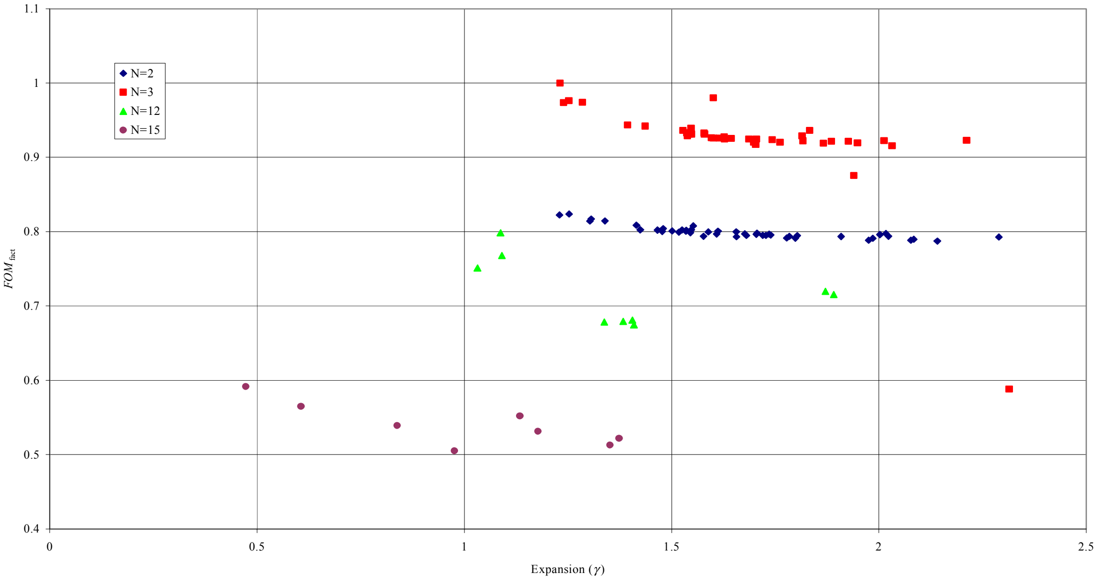

Figure 9. FOM_{fact} as a function of the optimized expansion values for the two-contraction factor case. The results for the many data runs are shown for $N = 2$, 3, 12, and 15. Similar results are obtained for the other dimensions.

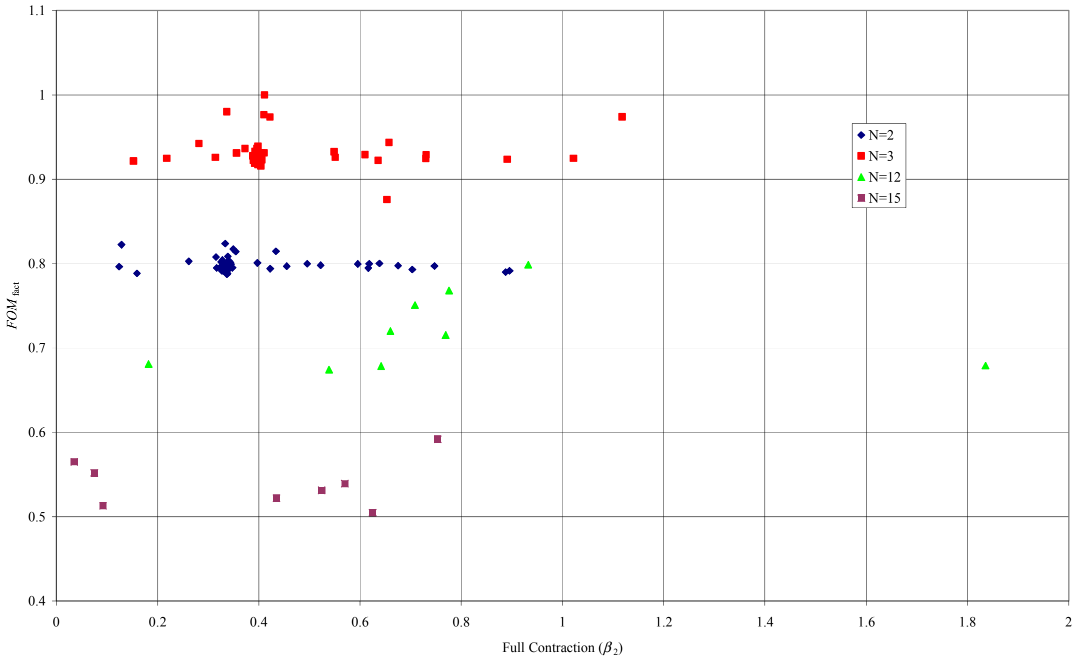

Figure 10. FOM_{fact} as a function of the optimized full contraction values for the two-contraction factor case. The results for the many data runs are shown for $N = 2$, 3, 12, and 15. Similar results are obtained for the other dimensions.

4.2 Illumination Optimization Results

The next step was to use the optimal simplex factor values in an optimization of an illumination system design. The chosen illumination system was a source placed in one aperture of a compound parabolic concentrator (CPC). The variables during the optimization were the radius of the disc source and the half angle of emission from this disc source (i.e., $N = 2$). The CPC was setup to have a source aperture acceptance half angle of 90° and a radius of 25.4 mm. Additionally, the output half angle of the CPC was 30°. The simplex routine was encoded within an *Advanced Systems Analysis Program* macro [8]. A random simplex was developed and the system was optimized with the altered factor values. The equivalent of two million random rays were traced from the source to the target plane in each iteration. In order to test the utility of the new simplex factors, the standard Nelder and Mead factor values with the same random simplex was done. The stopping condition for both cases was

$$\sigma^2(f) < \varepsilon_{tol}, \tag{7}$$

where

$$f = \frac{\Delta I_{PV}}{\eta} \tag{8}$$

and ε_{tol} is the prescribed tolerance, σ is the variance of the merit function, ΔI_{PV} is the peak-to-valley intensity value, and η is the transfer efficiency to the output aperture. The results are provided for both cases in Table 3, and intensity distribution plots for the altered case in Fig. 11 and the standard case in Fig. 12. As can be seen there is over a 25% improvement in iterations of the simplex algorithm with the use of the $N = 2$ factors from Table 2. This improvement is better than expected due to the results provided in Table 2; however, only one simplex optimization was performed and one is not necessarily only in the end game.

Table 3. Results for the standard and altered simplex factor optimizations of a CPC.

	r_s (mm)	θ_s (deg)	Stopping	*FOM*
Standard	25.27	89.14	1.213	66
Altered	24.40	87.33	1.199	47

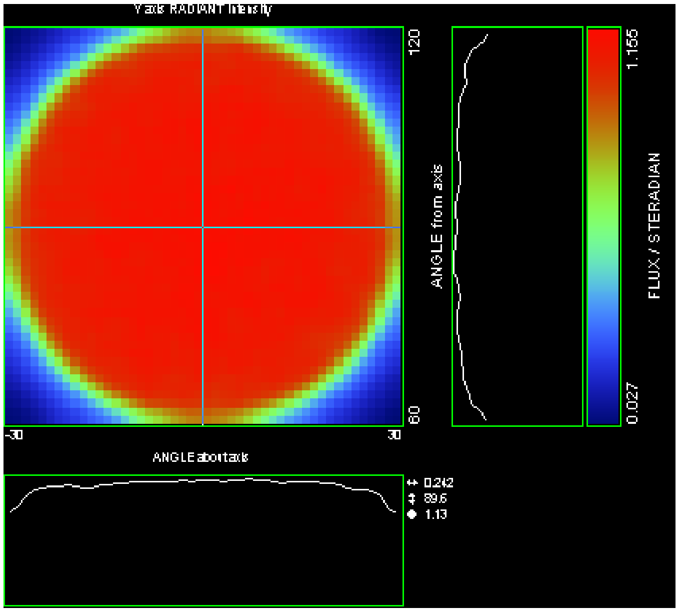

Figure 11. Intensity output distribution for the altered simplex case.

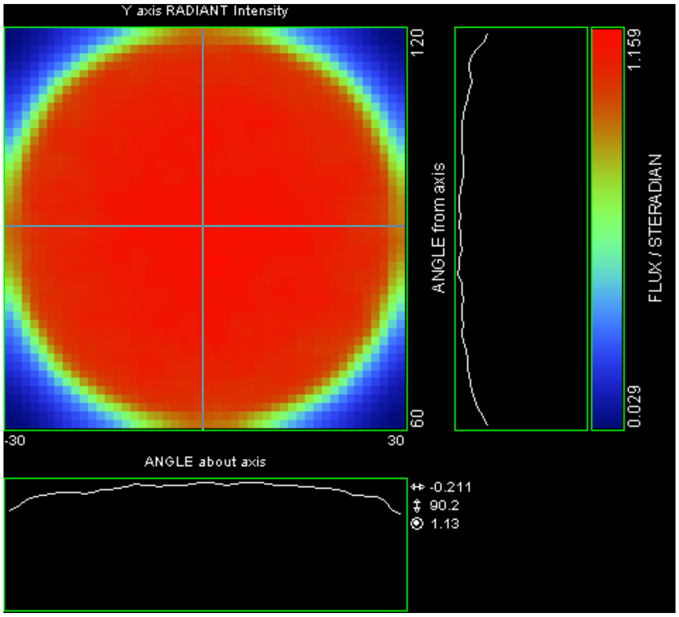

Figure 12. Intensity output distribution for the standard simplex case.

5. Conclusions

In the end game the expansion factor, α, is close to the original Nelder and Mead value of 1.0. It does increase as the order of the parabolic well increases. The 1D contraction factor, β_1, steadily increases with N, and plays the primary role in convergence in the end game. The expansion factor, γ, steadily decreases as N increases. This result indicates that this factor plays only a minor role in locating the minimum. Finally, the full contraction factor, β_2, is somewhat random. This result indicates that, as expected, this term plays virtually no role in the converging to the minimum. At $N = 5$ the altered simplex factors are the closest to the original Nelder and Mead values. The result is that speed of convergence is only 0.5% better with the altered factors. At other dimensional values, the speed of convergence is improved in the end game. For example near the extremes of the dimensional space investigate: $N = 2$ has a 21% improvement and $N = 14$ has a 35% improvement. Using these the optimized simplex values for N = 2, an illumination example was studied. In this example, the speed of convergence was improved by over 25%. Thus, a revised simplex method using these altered factor values will speed up convergence, especially in the end game.

6. Additional development

Additional research into improving the simplex method of optimization can be done. This additional work includes:

(1) The simplex method does not include any form of the derivative in its determination of the centroid or full contraction step. It is suggested that the inclusion of weighting terms related to the functional value for each point in the simplex can lead to an improvement in the convergence.
(2) Non-end game cases will be studied. The most suitable method to investigate these cases are by using functions from the optimization literature (e.g., Rosenbrock function) and optical design, in particular illumination, examples.
(3) Dynamic adjustment of the simplex parameters, such that the “start game” and “end game” aspects of the optimization can be employed to increase the speed of convergence.
(4) Multiprocessing methods such that separate processors investigate alternate paths concurrently. This inclusion will allow better local solutions to be located.

These additional items will be the focus of work in the future

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7. References

[1] W. Spendley, G. R. Hext, and F. R. Himsworth, “Sequential application of simplex designs in optimization and evolutionary operation,” *Technometrics* **4**, 441-461 (1962).
[2] J. A. Nelder and R. Mead, “A simplex method for function minimization,” *The Comp. Jnl* **7**, 308-313 (1965).
[3] William H. Press, Brian P. Flannery, Saul A. Teukolsky, and William T. Vetterling, *Numerical Recipes in Pascal: The Art of Scientific Computing* (Cambridge University Press, Cambridge, 1989), Chap. 10.4.
[4] Russell R. Barton and John S. Ivey, Jr., “Modifications of the Nelder-Mead simplex method for stochastic simulation response optimization,” in Proc. of the 1991 Winter Simulation Conf., Barry L. Nelson, W. David Kelton, and Gordon M. Clark, eds., pp. 945-953 (1991).
[5] Laura A. Yarbro and Stanley N. Deming, “Selection and preprocessing of factors for simplex optimization,” *Analytica Chimica Acta* **73**, 391-398 (1974).
[6] J. W. Akitt, “Function minimization using the Nelder and Mead simplex method with limited arithmetic precision: the self-regenerative simplex,” *The Comp. Jnl* **16**, 84-85 (1975).
[7] Giovanni P. Barabino, Giulio S. Barabino, Bruno Bianco, and Michele Marchesi, “A study on the performance of simplex methods for function minimization,” in Proc. of IEEE CH1511-5 (Institute of Electrical and Electronics Engineers, New York, 1980), pp. 1150-1153.
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